Roll No.							

24-MA-42

M.Sc. IV SEMESTER [MAIN/ATKT] EXAMINATION
JUNE - JULY 2024

MATHEMATICS
Paper - II
[Advanced Special Functions - II]

[Max. Marks : 75] *[Time : 3:00 Hrs.]* *[Min. Marks : 26]*

Note : Candidate should write his/her Roll Number at the prescribed space on the question paper.
Student should not write anything on question paper.
Attempt five questions. Each question carries an internal choice.
Each question carries **15 marks.**

Q. 1 a) Let $e^t \psi(x t) = \sum_{n=0}^{\infty} \sigma_n(x) t^n$ Then Prove that **(07 Marks)**

$\sigma_0'(x) = 0$ and $x \sigma_n'(x) - n \sigma_n(x) = -\sigma_{n-1}(x)$ for $n \geq 1$.

b) If $G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x) t^n$, then prove that $g_0'(x) = 0$ and for $n \geq 1$ **(08 Marks)**

$x g_n'(x) - n g_n(x) = g_{n-1}'(x)$

OR

a) $e^t \psi(x t) = \sum_{n=0}^{\infty} \sigma_n(x) t^n$, $\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n$. Then prove that for arbitrary c

$$(1-t)^{-c} F\left(\frac{x t}{1-t}\right) = \sum_{n=0}^{\infty} (c)_n \sigma_n(x) t^n \text{ where } F(u) = \sum_{n=0}^{\infty} (c)_n \gamma_n u^n$$ **(07 Marks)**

b) If $\{y_n\}$ be the polynomial defined by - **(08 Marks)**

$$A(t) e^{\frac{-x t}{1-t}} = \sum_{n=0}^{\infty} y_n(x) t^n$$

Then prove that

i) $y_0'(x) = 0$

ii) $y_n'(x) = y_{n-1}'(x) - y_{n-1}(x)$, $n \geq 1$

ii) $y_n'(x) = -\sum_{k=0}^{n-1} y_k(x)$, $n \geq 1$

P.T.O.

Q. 2 a) State and prove Rodrigue's formula for $P_n(x)$. **(08 Marks)**

b) Prove that **(07 Marks)**

$$(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}$$

OR

a) Show that **(08 Marks)**

$$P_n(x) = {}_2F_1 \left[\begin{matrix} -n, n+1 ; & \dfrac{1-x}{2} \\ 1 \;\; ; & \end{matrix} \right]$$

b) Prove that **(07 Marks)**

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} [x + \sqrt{(x^2-1)} \cos\phi]^n \, d\phi$$

Q. 3 a) Show that **(07 Marks)**

$$2x\, H_n(x) = 2\, n\, H_{n-1}(x) + H_{n+1}(x)$$

b) Prove that **(08 Marks)**

$$\sum_{n=0}^{\infty} \frac{H_{n+m}(x)\, t^n}{n\,!} = e^{2xt - t^2}\, H_m(x-t)$$

OR

a) Prove that $H_n(-x) = (-1)^n H_n(x)$ **(07 Marks)**

b) Show that for if m = n **(08 Marks)**

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx = 2^n \,.\, n\,!\, \sqrt{\pi}$$

Q. 4 a) Prove that **(07 Marks)**

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{m=0}^{n} L_m^{(\alpha)}(x)\, L_{n-m}^{(\beta)}(y)$$

b) Prove that **(08 Marks)**

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n\,!} D^n \{ e^{-x} x^{n+\alpha} \}$$

OR

a) Show that $\int_0^{\infty} e^{-x} x^{\alpha} L_m^{\alpha}(x)\, L_n^{(\alpha)}(x)\, dx = 0$ if $m \neq n$ for Re $\alpha > -1$ **(07 Marks)**

b) $\int_0^{\infty} e^{-x} x^{\alpha} [L_n^{\alpha}(x)]^2\, dx = \dfrac{\Gamma(1+\alpha+n)}{n\,!}$ for Re $\alpha > -1$ **(08 Marks)**

Cont. . . .

Q. 5 a) Show that **(08 Marks)**

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_{2n}}{n!\,(1+\alpha+\beta)_n}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,\ -\alpha-n\ ; & \dfrac{2}{1-x}\\ -\alpha-\beta-2n\ ; & \end{matrix}\right]$$

b) $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ **(07 Marks)**

OR

a) Prove that if $m \neq n$ **(08 Marks)**

$$\int_{-1}^{+1} (1-x)^{-\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\,.\,dx = 0$$

b) Show that **(07 Marks)**

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)\, t^n}{(1+\alpha)_x (1+\beta)_x} = {}_0F_1\left[\begin{matrix}- & ; & \dfrac{t(x-1)}{2}\\ 1+\alpha & ; & \end{matrix}\right]\ {}_0F_1\left[\begin{matrix}- & ; & \dfrac{t(x+1)}{2}\\ 1+\alpha; & & \end{matrix}\right]$$

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